On the Nonsingularity of Real Matrices

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Abstract. By exploiting the theory of linear inequalities, new bounds for the real eigenvalues of a real matrix are derived, along with sufficient conditions for matrix games to be completely mixed, for determinants to be positive, etc. The simple observation on which the derivation of new results and the unification of old results are based is that the typical conditions of diagonal dominance which insure the nonsingularity of matrices are essentially systems of linear inequalities on the rows of the matrices.

I. Introduction. Our purpose in this note is to derive and unify some old and new results on matrix games, and on bounds for eigenvalues, as simple consequences of the viewpoint of the theory of linear inequalities. We consider real matrices of order n, typically denoted by $A = (a_{ij})$. A well-known sufficient condition for such a matrix to be nonsingular is

$$(1.1) \qquad |a_{ii}| > \sum_{j \neq i} |a_{ij}|, \qquad i = 0, \cdots, n-1.$$

If we imagine that the matrix has been multiplied suitably by a diagonal matrix, in order that the new diagonal elements be nonnegative, then we may rewrite (1.1) as

$$(1.2) \qquad a_{ii} > \sum_{j \neq i} |a_{ij}|, \qquad i = 0, \cdots, n-1.$$

An alternative way of stating (1.2) is the following: Let M_i be the matrix with n rows and 2^{n-1} columns, where each column has a 1 in the ith row, and the other entries are $+1$ or -1 in all possible ways. Let the rows of A be denoted by $A_0', \cdots, A_{n-1}'$. Then (1.2) may be rewritten as

$$(1.3) \qquad A_i'M_i > 0, \qquad i = 0, \cdots, n-1.$$

Our object in this note is to characterize all possible sets of n matrices $M_0, \cdots, M_{n-1}$, such that (1.3) implies A is nonsingular (Theorem 1). As applications, we shall discover new classes of such matrices $\{M_i\}$, unifying material in [1], [2] and [3] on "completely mixed" games, new conditions for some matrices to have positive determinant, and new results on bounds for real eigenvalues of real matrices.

II. Main Theorem.

THEOREM 1. *Let $M_0, \cdots, M_{n-1}$ be n matrices, each with n rows, but with the number of columns of each matrix unspecified. Let $C_i = \{M_i x \mid x \geqq 0\}$. Assume that, for each i, C_i is a pointed cone (i.e., there exists u_i, $i = 1, \cdots, n$, such that $u_i'M_i > 0$). Then*

$$(2.1) \qquad (1.3) \textit{ implies } A \textit{ is nonsingular}$$

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if and only if

$$\bigcup(\pm C_i) = R^n. \tag{2.2}$$

Proof. Assuming (2.2), we must show that A is nonsingular. The proof is essentially the same as the standard proof [4] that (1.1) implies A nonsingular. If A is singular, there exists a nonzero vector y such that

$$Ay = 0. \tag{2.3}$$

By (2.2), y or $-y$ is contained in one of the cones C_i. Without loss of generality, assume y is in some C_i. Then, from (2.3), we have $0 = A_i'y = A_i'M_ix > 0$, since each coordinate of $A_i'M_i$ is positive (by (1.3)) and $x \geqq 0$, $x \neq 0$.

Conversely, assume (2.2) false, i.e., there exists a vector y such that, for each i,

$$y \notin C_i, \tag{2.4}$$

and

$$-y \notin C_i. \tag{2.5}$$

From (2.4), by the hyperplane separation theorem for cones, we infer that there exists a vector z_i such that

$$(y, z_i) < 0, \qquad z_i'M_i \geqq 0.$$

Since we know there exists u_i satisfying $u_i'M_i > 0$, we could replace z_i in the above inequalities by $z_i + \alpha_i u_i$ for α_i a small positive number, and obtain, for the new z_i,

$$(y, z_i) < 0, \qquad z_i'M_i > 0. \tag{2.6}$$

Similarly, from (2.5), we infer the existence of w_i such that

$$(y, w_i) > 0, \qquad w_i'M_i > 0. \tag{2.7}$$

From (2.6) and (2.7), we see that there exists a positive combination of z_i and w_i, orthogonal to y. Call that vector A_i. We then have

$$(y, A_i) = 0, \qquad A_i'M_i > 0. \tag{2.8}$$

In this manner, we construct a matrix A which is singular, yet satisfies (1.3).

III. Application. We now turn to the question of finding matrices $\{M_i\}$, other than those mentioned in the Introduction, for which (2.2) holds. Let us first describe a general method for discovering some classes $\{M_i\}$, and consider particular instances in the next section. Let u be any nonzero vector in R^n, and let $L = \{x \mid (x, u) = 0\}$. In L, choose n vectors $v_0, \cdots, v_{n-1}$ such that $x \in L$ implies x is a nonnegative combination of at most $n - 1$ of the vectors $\{v_1, \cdots, v_{n-1}\}$. (It is easy to find such a set of vectors; let $v_1, \cdots, v_{n-1}$ be a basis for L, and set $v_0 = -\sum_{i=1}^{n-1} v_i$.) It is clear that, if M_i has n columns consisting of u and all the vectors $\{v_0, \cdots, v_{n-1}\}$ except v_i, then $\bigcup C_i = \{x \mid (u, x) \geqq 0\}$. In other words, $\bigcup C_i$ is a closed half-space. Therefore, $\bigcup (\pm C_i) = R^n$.

As an illustration of the foregoing, we prove

THEOREM 2. *Let* $\beta_0, \beta_1, \cdots, \beta_{n-1}$ *be nonnegative, with* $\sum \beta_i = 1$, *and assume* g.c.d. $\{j \mid \beta_j > 0\}$ *is relatively prime to* n. *If* A *satisfies*

$$\text{(a)}\quad \begin{aligned} &\sum_j a_{ij} > 0, \; i = 0, \cdots, n-1, \quad \textit{and} \\ &\sum_j a_{ij}\beta_{-r+j} > a_{ir}, \quad i = 0, \cdots, n-1, r = 0, \cdots, n-1, r \neq i, \end{aligned}$$

or

$$\text{(b)}\quad \begin{aligned} &\sum_j a_{ij} > 0, \quad i = 0, \cdots, n-1, \quad \textit{and} \\ &\sum_j a_{ij}\beta_{-r+j} < a_{ir}, \quad i = 0, \cdots, n-1, r = 0, \cdots, n-1, r \neq i, \end{aligned}$$

or

$$\text{(c)}\quad \begin{aligned} &\sum_j a_{ij} < 0, \quad i = 0, \cdots, n-1, \quad \textit{and} \\ &\sum_j a_{ij}\beta_{-r+j} > a_{ir}, \quad i = 0, \cdots, n-1, r = 0, \cdots, n-1, r \neq i, \end{aligned}$$

or

$$\text{(d)}\quad \begin{aligned} &\sum_j a_{ij} < 0, \; i = 0, \cdots, n-1, \quad \textit{and} \\ &\sum_j a_{ij}\beta_{-r+j} < a_{ir}, \quad i = 0, \cdots, n-1, r = 0, \cdots, n-1, r \neq i, \end{aligned}$$

then $|A| \neq 0$. *Furthermore, the sign of* $|A|$ *is positive in case* (a), *negative in case* (c), $(-1)^{n-1}$ *in case* (b), $(-1)^n$ *in case* (d).

Proof. To prove the nonsingularity of A, we apply the discussion in the preceding paragraph. In cases (a) and (b), $u = (1, \cdots, 1)$. In cases (c) and (d), we use $-u$ in place of u. Let v_i, $i = 0, \cdots, n-1$, be the vector whose jth coordinate is $-\delta_{ij} + \beta_{j-i}$. One can show that any $n-1$ of the vectors $\{v_i\}$ are a basis of L, and the remaining vector is the negative of their sum. In cases (a) and (c), we use the vectors $\{v_i\}$. In cases (b) and (d), we use the vectors $\{-v_i\}$. It is then straightforward to verify the nonsingularity of A in each of the four cases.

To prove our statements about the signs of $|A|$, let us first treat case (a). It is clear that, for $\lambda \geqq 0$, the matrix $A + \lambda I$ satisfies the conditions of case (a). If λ is very large, then clearly $|A + \lambda I| > 0$. By the continuity of $|A + \lambda I|$ as a function of λ, if $|A| < 0$, there would have to be some positive λ, say λ_0, such that $|A + \lambda_0 I| = 0$, a contradiction. To treat case (d), we observe that, in that case, $-A$ satisfies the conditions of (a). Therefore, $|A| = (-1)^n |-A|$ has the sign $(-1)^n$.

In case (b), denote by J the matrix every entry of which is unity. Then $A + \lambda(J - I)$, for $\lambda \geqq 0$, satisfies the conditions of (b). But, for λ large and positive, $|A + \lambda(J - I)|$ has the same sign as determinant $|J - I|$, which has the sign $(-1)^{n-1}$. As before, the proof is completed by appealing to the continuity of $|A + \lambda(J - I)|$. Finally, case (c) is handled by observing that, in that instance, $-A$ is covered by (b).

THEOREM 3. *Let* A *satisfy any of the four conditions stated in Theorem* 2. *Then the* *system of equations* $\sum_i a_{ij}x_i = 1$ *has a (unique) solution in which each* $x_i > 0$ *in cases* (a) *and* (b), *each* $x_i < 0$ *in cases* (c) *and* (d).

Proof. We shall only treat case (a), the proofs of the other cases being similar. By virtue of what has gone before, all we need prove is that the determinant of the matrix obtained by replacing the ith row of A by the vector u is not zero, and has the same sign as $|A|$. Observe that the new matrix satisfies the conditions of case (a), except that the strong inequalities are replaced by weak inequalities. Since the determinant of a matrix is a continuous function of its entries, it follows that the determinant, if not zero, has the same sign as the determinant of A. Suppose y is a vector annihilated by the new matrix. Then, since y is orthogonal to u, $y \in L$. Therefore, y is a nonnegative combination of at most $n - 1$ of the vectors $\{v_i\}$. Suppose that the vector omitted in this nonnegative combination is v_k, $k \neq i$. Then, since A_k' makes a positive inner product with each vector in $\{v_i\}$ other than v_k, A_k' could not be orthogonal to y. If the vector omitted is v_i, replace y by $-y$. It cannot happen that, also for $-y$, the vector omitted is again v_i, for it is easy to show that the cone in L formed by any $n - 1$ of the v's is pointed. Thus, the new matrix cannot be singular, and our theorem is proven.

In the case when $\beta_{n-1} = 1$ (and all other β_i are zero), Theorems 1 and 2 include the results of [3]. In the case when all β_i are the same (i.e., $1/n$) our theorem gives another easy sufficient condition for a matrix game to be "completely mixed." These results dispose of the "sticky" example cited in [1] and [2].

IV. Bounds for Real Eigenvalues of Real Matrices. It is well known [4] that, from conditions on the coefficients of a matrix which insure nonsingularity, one may derive bounds for the eigenvalues of that matrix. In order to apply the foregoing results, it is useful for us to first state an alternative form of Theorem 2.

THEOREM 2′. *Let* $\beta_0, \beta_1, \cdots, \beta_{n-1}$ *satisfy the conditions of Theorem* 2. *Consider any subset* $S \subset \{0, 1, \cdots, n-1\}$, *and let* $\bar{S}$ *be the complementary set of indices. If inequalities* (a) *of Theorem* 2 *are satisfied for* $i \in S$, *and inequalities* (d) *are satisfied for* $i \in \bar{S}$, *then* A *is nonsingular. Similarly, if inequalities* (b) *are satisfied for* $i \in S$, *and inequalities* (c) *are satisfied for* $i \in \bar{S}$, *then* A *is nonsingular.*

Proof. Going back to Theorem 1, the only change we have made in Theorem 2 is to replace C_i, for $i \in \bar{S}$, by $-C_i$. By virtue of (2.2), this change does not affect the nonsingularity of A.

It is simplest for us to consider the case of Theorem 2 in which all β_i are $1/n$. (Conditions somewhat more complicated to state are inferrable when the β_i are any nonzero numbers, not necessarily the same.) Then the inequalities in each of (a), (b), (c), (d) become, respectively,

$$\text{(a)} \qquad \sum_j a_{ij} > 0, \qquad \sum_j a_{ij} > n \max_{j \neq i} a_{ij},$$

$$\text{(b)} \qquad \sum_j a_{ij} > 0, \qquad \sum_j a_{ij} < n \min_{j \neq i} a_{ij},$$

$$\text{(c)} \qquad \sum_j a_{ij} < 0, \qquad \sum_j a_{ij} > n \max_{j \neq i} a_{ij},$$

$$\text{(d)} \qquad \sum_j a_{ij} < 0, \qquad \sum_j a_{ij} < n \min_{j \neq i} a_{ij}.$$

We shall also use, for any real number a, the symbols a_+ and a_-, where

$$\begin{aligned} a_+ &= a && \text{if } a \geqq 0, \\ &= 0 && \text{if } a < 0, \\ a_- &= a && \text{if } a \leqq 0, \\ &= 0 && \text{if } a > 0. \end{aligned}$$

THEOREM 4. *For any real matrix A define*

$$P_i = \sum_j a_{ij} - \left(n \max_{j \neq i} a_{ij}\right)_+,$$

$$Q_i = \sum_j a_{ij} - \left(n \min_{j \neq i} a_{ij}\right)_-.$$

Then every real eigenvalue λ of A lies in the union of the closed intervals $[P_i, Q_i]$.

Furthermore, no eigenvalue of A satisfies the inequalities

$$\max_i \left\{ \sum_j a_{ij} - \left(n \min_{j \neq i} a_{ij}\right)_+ \right\} < \lambda < \min_i \left\{ \sum_j a_{ij} - \left(n \max_{j \neq i} a_{ij}\right)_- \right\}. \tag{4.1}$$

Proof. To prove the first part of the theorem, assume that the real eigenvalue λ of A does not lie in any of the intervals $[P_i, Q_i]$. Since the ith interval contains $\sum_j a_{ij}$, it follows that, for each i, $\lambda \neq \sum_j a_{ij}$. Let $S = \{i \mid \sum_j a_{ij} - \lambda > 0\}$, $\bar{S} = \{i \mid \sum_j a_{ij} - \lambda < 0\}$. Then, if $i \in S$, the ith row of $A - \lambda I$ satisfies inequality (a). If $i \in \bar{S}$, the ith row of $A - \lambda I$ satisfies inequality (d). By virtue of Theorem 2′, $A - \lambda I$ is nonsingular, contradicting the fact that λ is an eigenvalue of A.

To prove that no eigenvalue of A satisfies (4.1), let us first show that, if λ satisfies (4.1), then, for each i, $\lambda \neq \sum_j a_{ij}$. If not, then for some i, we have

$$\sum_j a_{ij} - \left(n \min_{j \neq i} a_{ij}\right)_+ < \sum_j a_{ij} < \sum_j a_{ij} - \left(n \max_{j \neq i} a_{ij}\right)_-.$$

If $\sum_j a_{ij}$ is subtracted from each expression, the resulting inequalities are inconsistent. Now, with S and $\bar{S}$ defined as before, we find that each row of $A - \lambda I$ satisfies either (b) or (c), and the argument is the same as before. In the case of the matrix $aI + bJ$, where J is the matrix every entry of which is unity, this theorem establishes that the only possible real eigenvalues of A are $a + nb$ and a, which is indeed the case.

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